Six Ports and Automatic Network Analysers -Two-dimensional error analysis

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Introduction

When trying to calculate uncertainty budgets for Ana's and Six Ports the question arises of "How accurate does the measurement of the real and imaginary parts have to be in order to achieve a particular confidence limit"? For NAMAS work a confidence limit of approximately 95.5% is used. When using an ANA or Six Port, each measurement point on the real and imaginary axis is made a number of times at each frequency and the centre of the "circle of results" is calculated. If the measurement is repeated the centre of the circle will almost certainly be in a different position.

Two-dimensional errors

The normal distribution of random errors deals with only one-dimensional (linear) error theory. The principles of error theory can be used advantageously to analyse results and to enable an estimate to be made for producing an uncertainty budget for a measurement procedure or a product specification.

To examine the accuracy of a point in a plane with respect to two axes, then some further error analysis is required. The linear error component of two and three dimensional positions can be analysed by applying the principles of normal linear error distribution

A two-dimensional error affecting a quantity is one defined by two random variables. For instance, the location of the point B in figure 1 is affected by two variables, direction and distance (or possibly by two distances only).

In the exaggerated diagram of figure 1 it becomes clearer. Since an error exists (either + or -) in the measured length of *AB* and in the measured direction of *AB*, then the near rectangle formed by 1, 2, 3, 4 would seem to define the location of the possible true position of *B*. Conversely, if we assume that B as plotted is the correct position of B, then the near rectangle 1, 2, 3, and 4 may be regarded as the limit for the positioning of *B*.

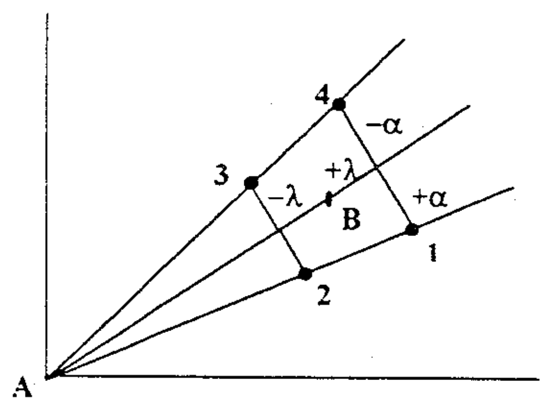

Figure 1

In figure 1 $+\alpha$ or $-\alpha$ is the error introduced by the error in the angle and $+\lambda$ or $-\lambda$ is the error in the length A to B

The Probability Ellipse

If we consider that $\pm\lambda$ is the error in the length and $\pm\alpha$ the error resulting from direction error, then assuming each is random and independent, the probability (linear) density distribution for each error is:

$$p_\lambda = \frac{1}{\sigma_\lambda\sqrt{2h}}\, e^{-\frac{\lambda^2}{2\sigma_\lambda^2}}$$

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The probability of each occurring simultaneously is the product of (p_λ) and (p_α), giving a two-dimensional probability density distribution:

$$(p_\lambda)(p_\alpha) = \frac{1}{(\sigma_\lambda)(\sigma_\alpha)(2h)}\, e^{-\left(\frac{\lambda^2}{2\sigma_\lambda^2}+\frac{\alpha^2}{2\sigma_\alpha^2}\right)}$$

This can be rewritten as:

$$(p_\lambda)(p_\alpha)(\sigma_\lambda)(\sigma_\alpha)(2h) = e^{-\frac{1}{2}\left(\frac{\lambda^2}{\sigma_\lambda^2}+\frac{\alpha^2}{\sigma_\alpha^2}\right)} \qquad \text{...(1)}$$

Taking Logs and rearranging gives:

$$-2\ln\left(p_\lambda p_\alpha \sigma_\lambda \sigma_\alpha (2h)\right) = \frac{\lambda^2}{\sigma_\lambda^2} + \frac{\alpha^2}{\sigma_\alpha^2}$$

Since, for given values of p_λ and p_α the left-hand side is a constant, then:

$$K^2 = \frac{\lambda^2}{\sigma_\lambda^2} + \frac{\alpha^2}{\sigma_\alpha^2}$$

This shows that, for values of $(p_\lambda)(p_\alpha)$ varying from 0 to ∞, a family of equal probability density ellipses will occur with axes $K\sigma_\lambda$ and $K\sigma_\alpha$.

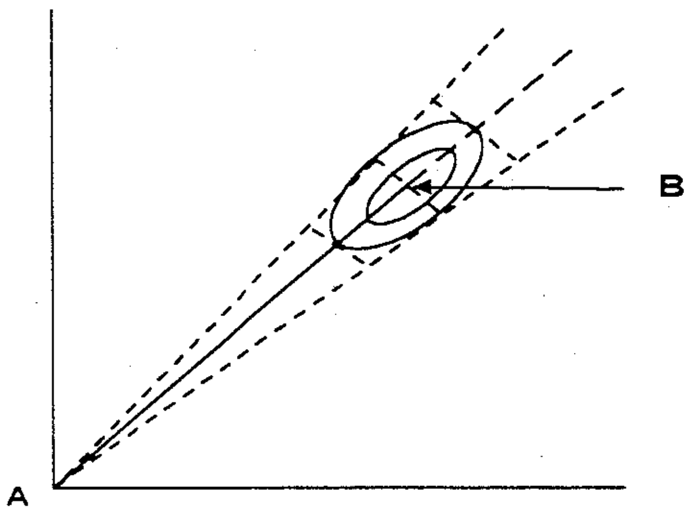
The Probability Circle

Assuming that $\alpha_\lambda = \sigma_\alpha$ the equation can be seen to be that of a circle (equal-axis ellipse), and by substituting and rearranging the terms, we obtain:

$$-2\sigma_\lambda^2 \ln\left[p_\lambda p_\alpha \sigma_\lambda^2 (2h)\right] = \lambda^2 + \alpha^2$$

The left-hand side of the equation is a constant, say, $(k_1 r)^2$ and now the circular form is clear.

$$(k_1 r)^2 = \lambda^2 + \alpha^2$$



Major axis $= \pm K\sigma_\lambda$ and Minor axis $= \pm K\sigma_\alpha$

Figure 2

So far we have been working with an error in length λ and an error in the direction α. By choosing measurement methods to keep σ_λ approximately equal to σ_α, the resulting error ellipse becomes approximately an error circle. The errors with respect to the co-ordinate axes will have exactly the same values, (See Figure 3) so that:

$$\sigma_\lambda = \sigma_\alpha = \sigma_x = \sigma_y$$

We may then conclude that the error circle will have a radius

$$k_2 = \sqrt{E_x^2 + E_y^2}$$

where E_x and E_y are any values of co-ordinate error.

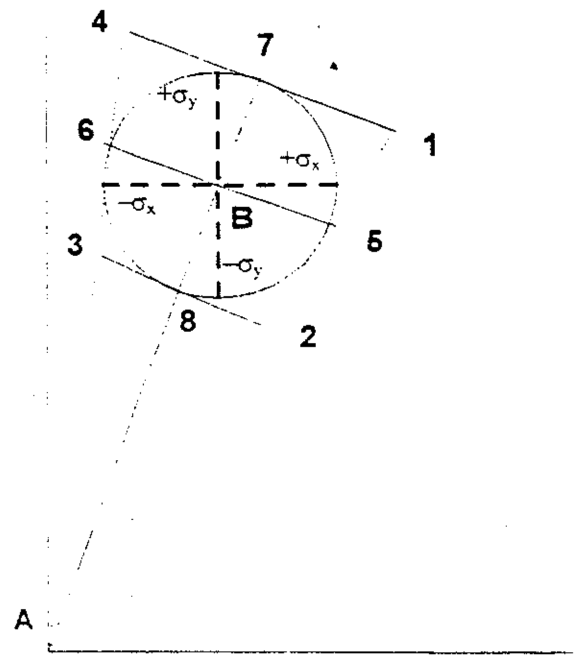


Figure 3

From figure 3

Length		Radius of Circle	
B to 5	$+\sigma_\alpha$	on line 5-6:	$\pm K\sigma_\alpha$
B to 6	$-\sigma_\alpha$	on line 7-8:	$\pm K\sigma_\lambda$
B to 7	$+\sigma_\lambda$	on horizontal axis	$\pm K\sigma_x$
B to 8	$-\sigma_\lambda$	on vertical axis:	$\pm K\sigma_y$

For convenience, assume the errors in the X and Y co-ordinates are x and y respectively. The previous elliptical probability density equation can now be written from equation (1) as :

$$p_x p_y \sigma_x \sigma_y (2h) = e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)}$$

By analogous reasoning as before , the probability density circle is given by :

$$\left(Kr^2\right) = x^2 + y^2$$

Elliptical (Circular) Error Evaluation

The probability density function when integrated gives the probability distribution function. The Probability of an ellipse is given by the distribution function:

$$P_{x,y} = 1 - e^{-\frac{K^2}{2}}$$

Solving this equation for various values of K yields the following values for probability percentages shown in Table 1.

Table 1

K	*Probability (P) %*
1.000	39.3
1.177	50.0
1.414	63.2
2.146	90.0
3.035	99.0
3.500	99.8

This means that when K = unity, for example, 39.3% of all errors in a circular distribution will be within the limits of the circular standard deviation σ_c..

Therefore when K = 1.0000, the axes of the ellipse are $1.0000\sigma_x$, and $1.0000\sigma_y$ (giving a circle of radius $\sigma_x = \sigma_y$) and that there is a 39.3% probability that the actual position errors in x and y will fall simultaneously within that circle.

Increasing the diameter of the circle to $3.5000\sigma_c$ (= $3.5000\sigma_x$ = $3.5000\sigma_y$) will give a 99.8% probability that both the x and the y error will fall within the circle.

So for a set of readings on say a network analyser if we wish to keep the result within the circular limits for a given probability we must use consistent real and imaginary values in terms of the units.

For example if we need to keep the position of B within a circle such that the probability is 99.8% then we know that Ex and Ey are $3.5\sigma_x$ and $3.5\ \sigma_y$ then :

$$0.005 = \pm 0.005/3.5 = 0.0014$$

Table 2 shows the values on the real and imaginary axis needed to maintain the location of B to be within ±0.005 for different values of K

Table 2		
K	*Probability (P) %*	*for ±0.005 in B*
1.000	39.3	0.0050
1.177	50.0	0.0042
1.414	63.2	0.0035
2.146	90.0	0.0023
2.45	95.0	0.0020
2.49	95.5	0.0020
3.035	99.0	0.0016
3.500	99.8	0.0014

If we require that B is within ±0.005 with a probability of 95.5% then we need the values on the x & y axis to be within ±0.002.

Reference

Engineering Measurements. by B.Austin Barry FSC
John Wiley & Sons Inc.