2 Port Gain and Stability

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Abstract

The fundamental requirements for stability of a 2 port network is described. This is investigated by way of device gain and the concept of gain circles, using the s-parameter formulation. The requirements in terms of a device's s-parameters are described. Those in circulation for a long time are confirmed as valid, as well as a more recent and concise single form.

Introduction

The constraints on a device, in terms of its s-parameters, that determine whether a 2 port is unconditionally or conditionally stable have been elaborated for quite some time. The s-parameter formulation appears for example in a paper by Kurokawa [1]. Unfortunately, the paper introduces an error which has been repeated in the literature. A more popular approach is described in a paper by Bodway [2], which addresses the question of device stability from a slightly different perspective. The results are sound, but the process to arrive at them are not clear.

Much later, the stability problem was addressed by Edwards [3], and a single stability criterion was shown to be sufficient. His paper on the subject also addressed the error in the Kurokawa paper.

The contribution of this paper is to demonstrate methodically how the question of 2 port stability can be determined. All mathematical steps are described in the derivation of key equations. The paper also introduces a new analytic device that improves the clarity of the arguments presented.

The Meaning of Stability

The issue of stability can be illustrated by a one port system as shown in Fig. 1.

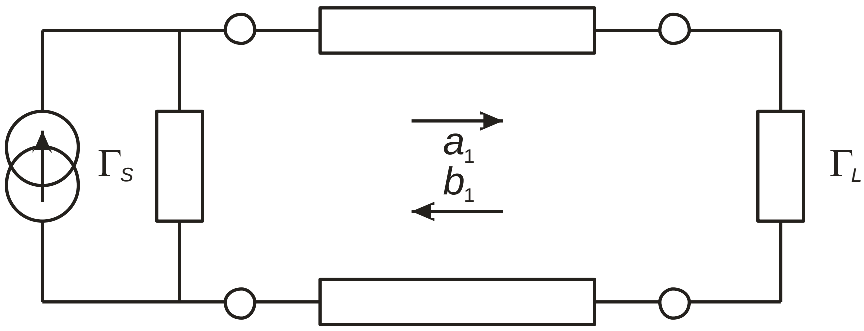

Figure 1. One Port System

The diagram illustrates a source connected to a load via a short transmission line, where:

Γ_S, Γ_L are the source and load reflection coefficients respectively

a_1, b_1 are the forward and reflected travelling waves respectively

The relationship of the load with forward and reflected waves is given mathematically as:

$$\Gamma_L = \frac{b_1}{a_1} \tag{1}$$

In order for existing waves to reduce with time we require:

$$|\Gamma_S \Gamma_L| < 1 \tag{2}$$

For waves to increase with time, we require:

$$|\Gamma_S \Gamma_L| > 1 \tag{3}$$

If we limit the source reflection coefficient such that its magnitude is no greater than 1, then stability requires the load reflection also has a magnitude no greater than 1.

This condition may also be applied to the impedance of the load. We have:

$$Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L} \tag{4}$$

In this analysis Z_0 will be assumed to be a purely real quantity. (4) can be manipulated into its real and imaginary parts, thus:

$$Z_L = Z_0 \frac{(1+\Gamma_L)(1-\Gamma_L^*)}{(1-\Gamma_L)(1-\Gamma_L^*)} = Z_0 \left(\frac{1-|\Gamma_L|^2}{|1-\Gamma_L|^2} + \frac{\Gamma_L - \Gamma_L^*}{|1-\Gamma_L|^2} \right) \tag{5}$$

The first term in brackets, being real, is positive for $|\Gamma_L| < 1$ and negative for $|\Gamma_L| > 1$.

A negative real impedance ($|\Gamma_L| > 1$) cannot absorb power, it can only generate it. Any source connected to a negative real impedance changes from being a generator to a motor!

Returning to the system of Fig. 1, the condition given by (3) is an undesirable condition for an amplifier, but a necessity for an oscillator. The inequality $|\Gamma_L| < 1$ gives rise to a condition known as *unconditional stability*, as no choice of passive source is able to generate ever increasing wave amplitude, whereas the inequality $|\Gamma_L| > 1$ is known as *conditional stability*, as a suitable choice of passive source can generate waves of ever increasing amplitude between it and the load.

Stability of a 2-Port

When we come to a 2-port, the reflection at one port is modified by the termination at the other. We are interested in whether the reflection coefficient of either port has a magnitude greater than 1, given any passive termination at the other. The 2-port under investigation is depicted in Fig. 2.

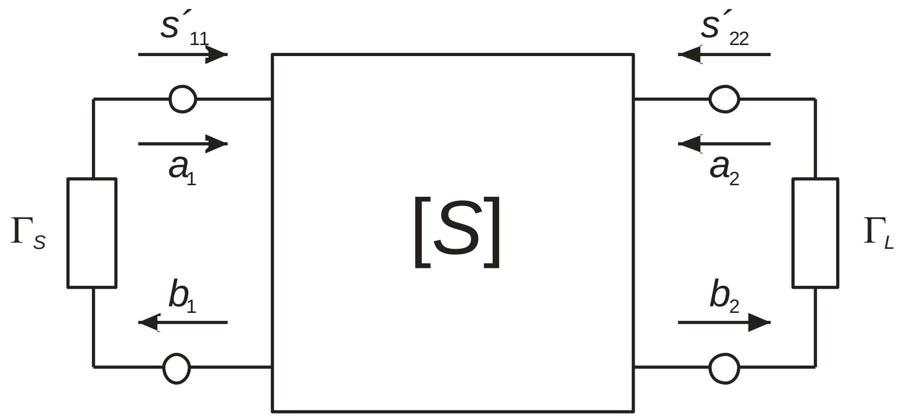

Figure 2. 2-Port Circuit

Either termination may also have a source generator associated with it. The 2-port is characterised by the matrix equation:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{6}$$

For excitation at the source only, the output waves are related by:

$$\Gamma_L = \frac{a_2}{b_2}$$

Using the above to substitute for b_2 into the second of (6), we can eliminate a_2 from the first equation and rearrange for b_1/a_1 giving:

$$s'_{11} = \frac{b_1}{a_1} = s_{11} + \frac{s_{12}s_{21}}{1-s_{22}\Gamma_L} = \frac{s_{11}-\Delta\Gamma_L}{1-s_{22}\Gamma_L} \qquad (7)$$

Where:
$$\Delta = s_{11}s_{22} - s_{12}s_{21} \qquad (8)$$

The use of the determinant of the s-matrix proves useful in condensing equations derived subsequently.

Similarly for the output we have:

$$s'_{22} = \frac{b_2}{a_2} = s_{22} + \frac{s_{12}s_{21}}{1-s_{11}\Gamma_S} = \frac{s_{22}-\Delta\Gamma_S}{1-s_{11}\Gamma_S} \qquad (9)$$

At this stage, we must suppose that two criteria must be met in order to assure unconditional stability. Firstly, any passive load impedance must not generate an input reflection coefficient greater than 1, and secondly, any passive source impedance must not generate an output reflection coefficient greater than 1.

The inquiry into stability is aided by considering device gain.

Available Gain

We shall use the concept of *available gain*, which is defined as the power available from the device divided by the power available from the source.

Available power is attained with a conjugate match of impedances. So for example, if the source has an impedance of $R + jX$, then the load needs to have an impedance of $R - jX$. Alternatively, we may express the source and load in terms of reflection coefficient, where likewise they are conjugates. With the definition of available gain as above, the load is conjugately matched to the output of the device. As the load is defined by the device parameters and source reflection coefficient, available gain is independent of load.

Available gain is applicable for example to low noise device design, where source is optimised for noise and gain is maximised using a conjugate output match.

We now need an expression for the available gain of the source, given the known quantities of its reflection coefficient and input side forward and reflected power waves. See the flow diagram of Fig. 3.

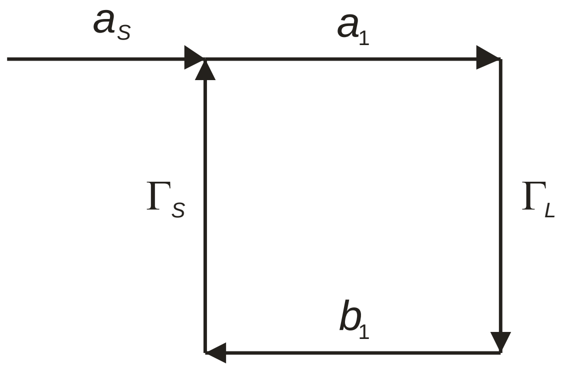

Figure 3. Model of Source with Load

The governing equations for the flow diagram of Fig. 3 are as follows:

$$a_1 = a_S + \Gamma_S b_1 \qquad (10)$$

$$b_1 = \Gamma_L a_1$$

Let $a_1 = a_m$ and $b_1 = b_m$ in the matched condition $\Gamma_L = \Gamma_S^*$. Thus:

$$b_m = \Gamma_S^* a_m$$

$$a_m = a_S + \Gamma_S b_m$$
$$= a_S + |\Gamma_S|^2 a_m$$

This gives us:

$$a_m = \frac{a_S}{1-|\Gamma_S|^2}$$

$$b_m = \frac{\Gamma_S^* a_S}{1-|\Gamma_S|^2}$$

Power delivered in the matched condition is given by:

$$|a_m|^2 - |b_m|^2 = \frac{|a_S|^2}{(1-|\Gamma_S|^2)^2} - \frac{|\Gamma_S a_S|^2}{(1-|\Gamma_S|^2)^2} = \frac{|a_S|^2}{1-|\Gamma_S|^2}$$

Substituting for a_S using (10) gives us for the power available from the source P_{Sav}:

$$P_{Sav} = \frac{|a_1 - \Gamma_S b_1|^2}{1-|\Gamma_S|^2} \quad (11)$$

Similarly, the power available from the output of the 2-port P_{2av} is given by:

$$P_{2av} = \frac{|b_2 - s'_{22} a_2|^2}{1-|s'_{22}|^2} \quad (12)$$

Available gain G_a is the ratio of (12) to (11), and substituting for b_1 and b_2 using (6) and s'_{22} using (9), the ratio simplifies to:

$$G_a = \frac{|s_{21}|^2 (1-|\Gamma_S|^2)}{|1-s_{11}\Gamma_S|^2 - |s_{22} - \Delta\Gamma_S|^2} \quad (13)$$

Note available gain is always zero when the source reflection coefficient is 1, because in this case available power from the source is infinite.

Inspection of the denominator of the above equation suggests that for appropriate choice of source reflection coefficient, available gain may be infinite or even negative. If the source is passive, then the conjugate matched load has a reflection coefficient greater than 1, so the device can be made unstable.

The enquiry turns on whether a passive source reflection coefficient of any phase and magnitude can generate an infinite or negative available gain, which brings us to the topic of gain circles.

<u>Available Gain Circles</u>

Let us use a modified version of available gain, the gain function g_a, defined as:

$$g_a = \frac{G_a}{|s_{21}|^2} = \frac{1-|\Gamma_S|^2}{|1-s_{11}\Gamma_S|^2 - |s_{22} - \Delta\Gamma_S|^2} \quad (14)$$

Now, recall the general equation of a circle in the complex plane, given by:

$$|Z - Z_C| = R \quad (15)$$

In (15), Z is the complex variable, Z_C is the complex centre of the circle and R is the real radius of the circle. From (15), we further develop as below:

$$|Z - Z_C|^2 = R^2$$

$$(Z - Z_C)(Z^* - Z_C^*) = R^2$$

$$|Z|^2 - Z_C^* Z - Z_C Z^* + |Z_C|^2 = R^2 \quad (16)$$

Expanding the magnitude expressions in the denominator of (14) into a complex function and its conjugate, the equation can subsequently be cast into the form of (16) by routine algebra. The equation circle of the form of (15) is then evident on inspection, which is:

$$\left|\Gamma_S - \frac{g_a(s_{11}^* - s_{22}\Delta^*)}{1+g_a(|s_{11}|^2-|\Delta|^2)}\right| = \frac{\sqrt{1-g_a(1-|s_{11}|^2-|s_{22}|^2+|\Delta|^2)+g_a^2|s_{12}s_{21}|^2}}{1+g_a(|s_{11}|^2-|\Delta|^2)} \qquad (17)$$

The sign of the square is chosen to yield a positive radius value. If the expression under the square root sign is negative, no real radius can be realised and that available gain value doesn't exist. Putting the gain function to zero defines the unit circle. If the gain function tend towards infinity, we get the *source stability circle* whose parameters upon inspection are:

$$C_S = \frac{s_{11}^* - s_{22}\Delta^*}{|s_{11}|^2-|\Delta|^2} \qquad (18)$$

$$R_S = \frac{|s_{12}s_{21}|}{|s_{11}|^2-|\Delta|^2} \qquad (19)$$

Available gain changes sign at the boundaries of the unit circle and source stability circle, dividing the source reflection plane into regions where the function g_a is positive and negative.

We require the available gain function to be positive within all the unit circle, which leads to two possibilities of source stability circle. These are shown in Fig. 4 (The Smith Chart is for identification purposes, we don't require particular knowledge of impedance).

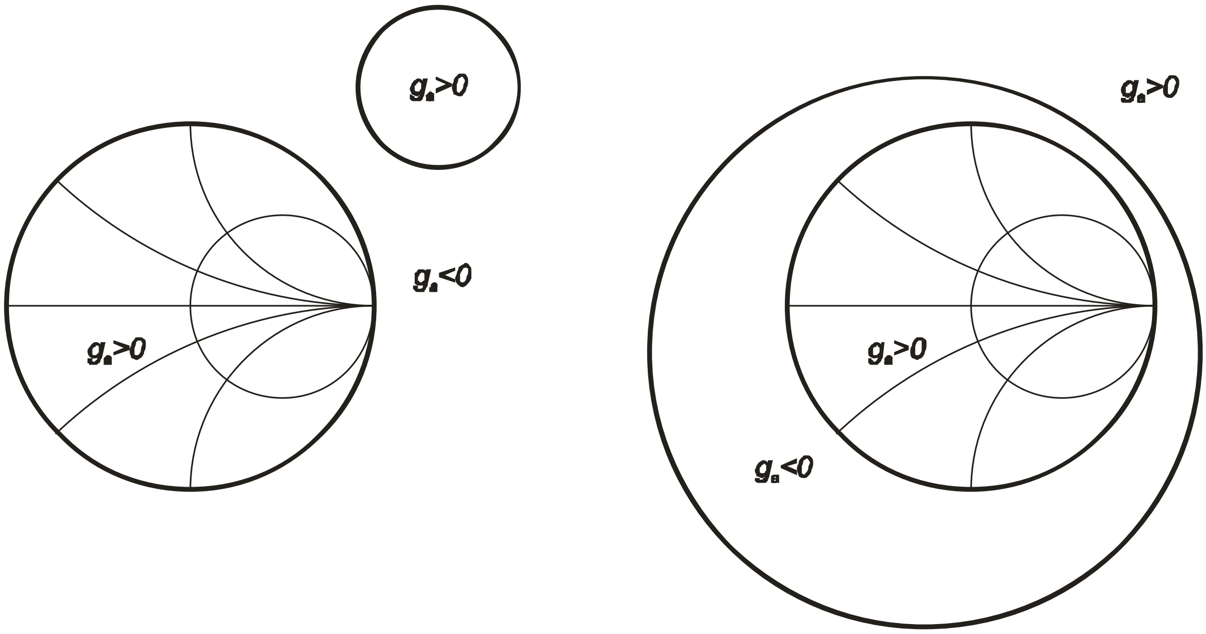

Figure 4. Unconditional Stability Circles

The two stability cases above may be thought to amount to the same thing, if the complex plane is transformed via a stereographic projection to the surface of a sphere.

There are a number of possibilities for conditional stability. For example, either case in Fig. 4, with gain polarities reversed. In order to assure stability, it is not sufficient simply to show the stability circle is outside the unit circle.

Two further cases of conditional stability are shown in Fig. 5.

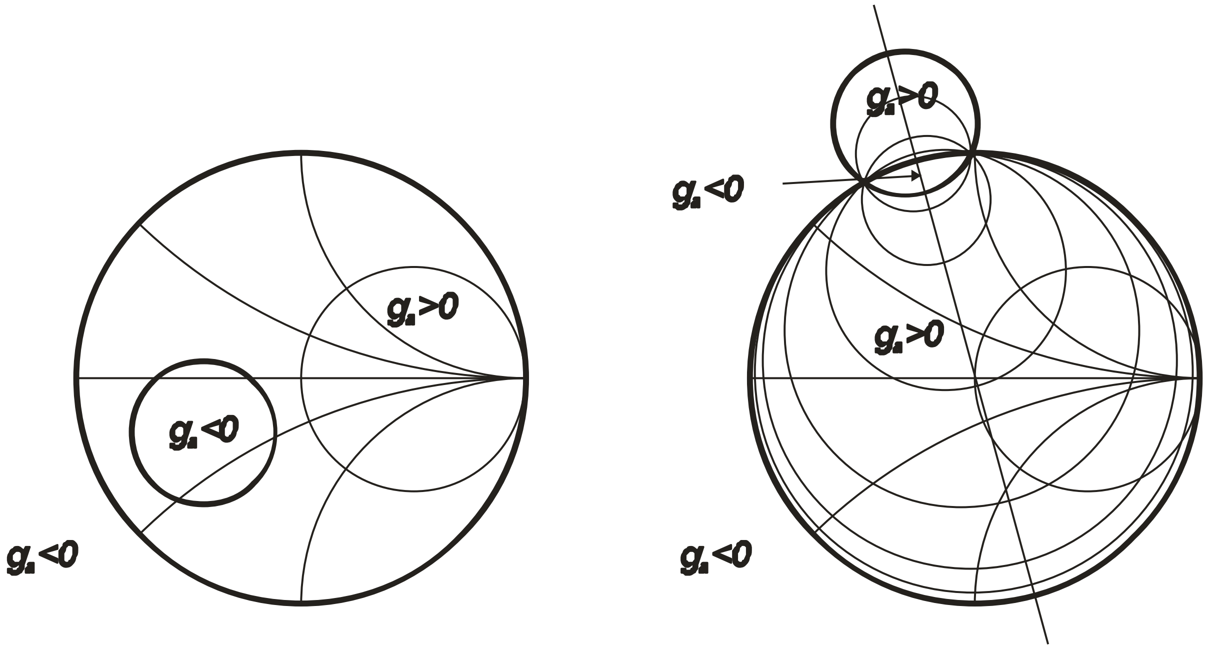

Figure 5. Conditional Stability Circles

If any part of the stability circle lies within the unit circle, then available gain changes polarity at the boundary, leading to a negative region within the unit circle. Note that the sign of available gain function may be opposite in all regions for either of the illustrations of Fig. 5.

The right hand diagram of Fig. 5 is the usual case for typical conditionally stable devices. There is an ambiguity in available gain at the points where the stability circle crosses the unit circle. The stability circle indicates available output power is infinite, whilst the unit circle indicates available source power is infinite, so the ratio becomes indeterminate. In this case, all gain circles (which otherwise never cross) pass through the same two intersection points.

Further points of interest for gain circles is where the radius is zero (Quantity under the square root sign is zero in (17)). The points are either minimum or maximum gain. Solving we have:

$$g_{am} = \frac{1-|s_{11}|^2-|s_{22}|^2+|\Delta|^2 \pm \sqrt{(1-|s_{11}|^2-|s_{22}|^2+|\Delta|^2)^2-4|s_{12}s_{21}|^2}}{2|s_{12}s_{21}|^2}$$

$$= \frac{K \pm \sqrt{K^2-1}}{|s_{12}s_{21}|} \qquad (20)$$

K is the *Rollett stability factor* [4], defined as:

$$K = \frac{1-|s_{11}|^2-|s_{22}|^2+|\Delta|^2}{2|s_{12}s_{21}|} \qquad (21)$$

For an unconditional stability solution, the available gain function has to have a positive maximum value in the unit circle, Hence $K > 1$. The maximum has to be less than the minimum (to avoid more than one gain circle for a given gain), so take the negative sign in (20).

We can't be sure simply because $K > 1$ that the device is stable, take for example the left hand diagram of Fig. 5 with gain polarities reversed, where the stability circle lies inside the unit circle and the gain function is positive within it. As the gain function is negative in the eccentric annulus between, the device is potentially unstable.

We can establish then that a device is stable by showing the stability circle lies outside the unit circle and available gain is positive for an arbitrary point inside the unit circle (the origin would seem to be a sensible point). However, the procedure of identifying the stability circle and making an evaluation at the centre is rather cumbersome. A more direct approach would be welcome, and this is discussed in the next section.

Available Gain Function along Gain Circle Centre Line

In (17) the centre varies with gain function, remains on a straight line (Indicated by the line in the right hand example of Fig. 5). We may examine gain function along the line (the analytic device referred to in the Introduction) by substituting the following variable for source reflection into (14):

$$\Gamma_S = \frac{(s_{11}^* - s_{22}\Delta^*)r}{|s_{11}^* - s_{22}\Delta^*|} \tag{22}$$

The result is:

$$g_{aC} = \frac{1 - r^2}{(|s_{11}|^2 - |\Delta|^2)r^2 - 2|s_{11}^* - s_{22}\Delta^*|r + 1 - |s_{22}|^2} \tag{23}$$

(23) tells us the value of the gain function along the gain circle centre line at a distance *r* from the origin. It passes twice through every possible gain circle, representing all possible values of the gain function. Thus thee analysis can be reduced to a two-dimensional graph.

From (23), we can clearly see that the gain function equals zero when $r = \pm 1$, the points where it crosses the unit circle. It is clear too, by being a bi-quadratic expression, for a given value of gain function only two solutions to *r* exist, so gain circles are unique.

The previous section determined the value of the gain function at its maximum and minimum. We can use (23) to find their position, by taking the differential and equating to zero for the turning points. Differentiating (22) we get:

$$\frac{dg_{aC}}{dr} = 2\frac{|s_{11}^* - s_{22}\Delta^*|r^2 - (1 + |s_{11}|^2 - |s_{22}|^2 - |\Delta|^2)r + |s_{11}^* - s_{22}\Delta^*|}{\{(|s_{11}|^2 - |\Delta|^2)r^2 - 2|s_{11}^* - s_{22}\Delta^*|r + 1 - |s_{22}|^2\}^2} \tag{24}$$

It is convenient to put [2]:

$$B_1 = 1 + |s_{11}|^2 - |s_{22}|^2 - |\Delta|^2 \tag{25}$$

$$C_1 = s_{11} - s_{22}^*\Delta \tag{26}$$

The solutions for the turning points are then given by:

$$r_{m1,2} = \frac{B_1 \pm \sqrt{B_1^2 - 4|C_1|^2}}{2|C_1|} \tag{27}$$

The product of the two solution is 1, the importance of which is that the turning points are on the same side of the origin, one being inside the unit circle and the other outside. The sign of the solutions can be determined from the sign of B_1.

The actual complex locations of the maximum and minimum points can be determined by substituting the solution above into (22), which become:

$$\Gamma_{Sm1,2} = C_1^* \frac{B_1 \pm \sqrt{B_1^2 - 4|C_1|^2}}{2|C_1|^2} \tag{28}$$

For real solutions, the quantity inside the square root sign must be positive. If it is negative, the stability circle intersects the unit circle and the solutions to (28) identify the crossing points.

By inspection, (24) is definitely positive at the origin, which means the gain function is increasing.

We now have the information required to establish the full conditions for stability. This may be illustrated by two examples.

Firstly, let us take the case of an unconditionally stable parameter matrix given by:

$$S = \begin{bmatrix} 0.679\angle 135.4° & 0.028\angle -31.8° \\ 1.894\angle -20.9° & 0.78\angle -127.6° \end{bmatrix}$$

The gain function along the gain circle centre line is from (23):

$$g_{aC} = \frac{1-r^2}{0.206r^2 - 0.577r + 0.392}$$

Evaluation of (21) yields *K* = 1.756. As *K* > 1, we know the stability circle doesn't intersect the unit circle and the maximum and minimum gain solutions are both positive. Plotting g_{aC} over the range *r* = -1 to 1, the result is as shown in Fig. 6.

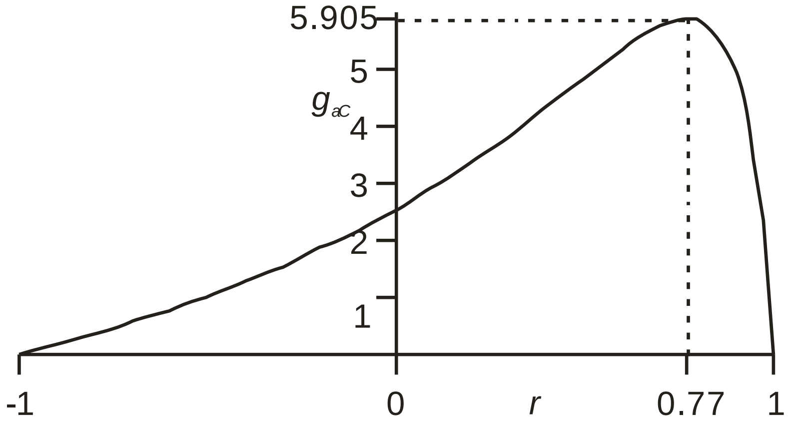

Figure 6. Unconditionally Stable Gain Function

In Fig. 6, the plot shows the characteristics common to all gain function plots, in that it is zero for *r* = ±1 and is positive going when *r* = 0. As *K* > 1, any maximum or minimum has a positive value. In this particular case, the stability circle is outside the unit circle, and these facts constrain the plot to exhibit a maximum in the *r* = 0 to 1 range.

Consider now a conditionally stable parameter matrix given by:

$$S = \begin{bmatrix} 0.9\angle -80° & 0.2\angle 30° \\ 8\angle 90° & 0.8\angle -35° \end{bmatrix}$$

Once again, *K* > 1 at a value of 1.234, however plotting the gain function along the same range gives the result as in Fig. 7.

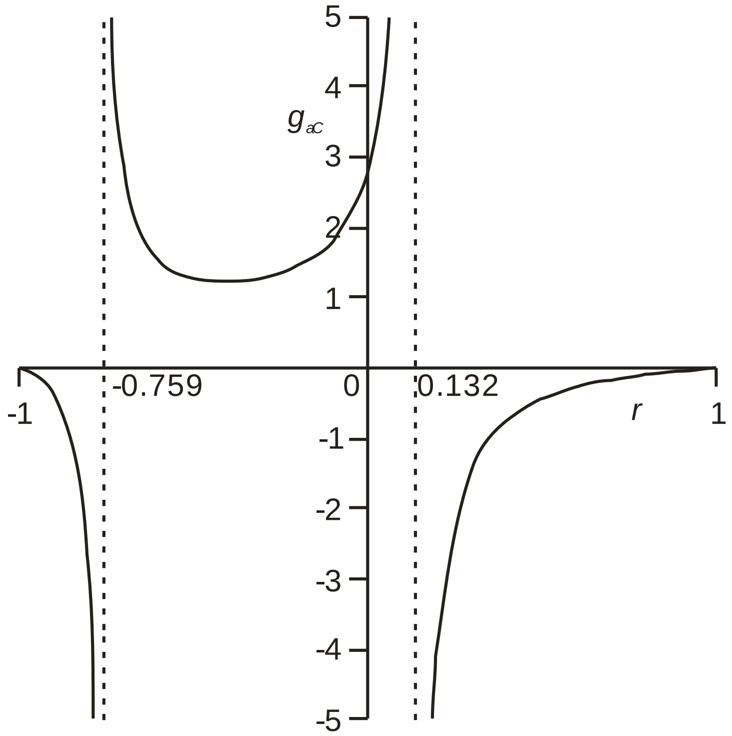

Figure 7. Conditionally Stable Gain Function

The infinite points of available gain and the negative values in the range r = -1 to +1 indicate the conditional stability. Available gain is zero where $r = \pm 1$ and is positive going where r = 0. Indeed it is positive going from the minimum point to r = 1 and negative going below that. The key difference between the two cases, unconditional and conditional stability, is that the turning point must be negative for the conditional stability case and positive for the unconditional stability case. Examining (27), this means B_1 must be positive rather than negative for unconditional stability.

We have finally reached the goal of our analysis, unconditional stability criteria are:

$$K > 1 \qquad\qquad B_1 > 0$$

Maximum Available Gain

The maximum value of the gain function in the unit circle occurs where input and output are simultaneously matched. The available gain definition already ensures an output match, but the presence of a maximum requires the input to be matched as well. Its position within the unit circle means the source reflection coefficient needs the negative sign in (28). The value of the gain function requires the negative sign in (20), as being a maximum it must be less than the minimum. Substituting this result back into (14) yields the maximum available gain given by:

$$G_{MAX} = \left|\frac{s_{21}}{s_{12}}\right| \left(K - \sqrt{K^2 - 1}\right) \tag{29}$$

Single Stability Criterion

The analysis so far has identified stability in terms of two criteria ($K > 1$, $B_1 > 0$). It is possible to reduce to a single stability criterion. To show this, we need to examine the points where the stability circle crosses the line of gain circle centres. We are most interested in the point nearest to the origin.

Having defined the polarity of r by (22), we can use the centre and radius information in (18) and (19) to determine the values of r where available gain function is infinite, and these become:

$$r_{n,f} = \frac{|s_{11}^* - s_{22}\Delta^*| \pm |s_{12}s_{21}|}{|s_{11}|^2 - |\Delta|^2} \quad (30)$$

In (30), the *n*, *f* subscripts are taken to indicate near (negative option) and far (positive option) respectively. We need at this point to examine the first magnitude expression in the numerator of (30). We can show that:

$$|s_{11}^* - s_{22}\Delta^*|^2 = (1 - |s_{22}|^2)(|s_{11}|^2 - |\Delta|^2) + |s_{12}s_{21}|^2 \quad (31)$$

((31) may be confirmed for example by evaluating (30) from the infinite solutions to (23).)

(30) presents a problem if the denominator is zero, as by (31), so is the numerator in the negative option and the result is indeterminate. However, if we take the product of the two solutions, we get:

$$r_n \times r_f = \frac{|s_{11}^* - s_{22}\Delta^*|^2 - |s_{12}s_{21}|^2}{(|s_{11}|^2 - |\Delta|^2)^2} = \frac{1 - |s_{22}|^2}{|s_{11}|^2 - |\Delta|^2} \text{ (using (31))}$$

Then we have:

$$r_n = \frac{r_n \times r_f}{r_f} = \frac{1 - |s_{22}|^2}{|s_{11}^* - s_{22}\Delta^*| + |s_{12}s_{21}|} \quad (32)$$

In order to assure stability, we need at least to know that the nearest point of the stability circle to the origin is outside the unit circle. This means $|r_n| > 1$. However, we can see from (32) that negative values of this point are not permitted, because the denominator being always positive, we need $|s_{22}| < 1$ for stability. A sufficient single criterion is therefore given by [3]:

$$\mu' = \frac{1 - |s_{22}|^2}{|s_{11}^* - s_{22}\Delta^*| + |s_{12}s_{21}|} > 1 \quad (33)$$

That the closest point of the stability circle is in the positive *r* direction is betrayed by Fig. 6. The slope of the plot is already much greater when $r = +1$ than it is when $r = -1$.

The result in (33) may be expressed in reverse port order, giving the alternative single stability criterion as follows:

$$\mu = \frac{1 - |s_{11}|^2}{|s_{22}^* - s_{11}\Delta^*| + |s_{12}s_{21}|} > 1 \quad (34)$$

Conclusions

Two alternative stability criteria have been presented. One which involves the criteria pair of $K > 1$, $B_1 > 0$ has the advantage that both parameters have other uses; *K* is required in the calculation of maximum available gain and B_1 in the calculation of simultaneous conjugate match.

The single stability criterion has the advantage of a more concise form.

References

[1] Kurokawa K., “Power Wave and the Scattering Matrix”, *IEEE Trans. Microwave Theory and Techniques*, Vol. MTT-13, No. 3, Mar. 1965, pp. 194-202.

[2] Bodway G. E., “Two Port Power Flow Analysis Using Generalized Scattering Parameters”, *Microwave J.*, May 1967, pp. 61-69.

[3] Edwards M. L., Sinsky J. H., “A New Criterion for Linear 2-Port Stability Using a Single Geometrically Derived Parameter”, *IEEE Trans. Microwave Theory and Techniques*, Vol. MTT-40, No. 12, Dec. 1992, pp. 2303-2310.

[4] Rollett J. M., “Stability and Power Gain Invariants of Linear Two-Ports”, *IRE Trans. Circuit Theory*, Vol CT-9, Mar. 1962, pp. 29-32.